# Thin-Shell Approach for Elastic Wave Propagation in a Pipe with Liquid

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This paper presents the validity and limitation of the thin-shell approach for the analysis of elastic wave propagation in a pipe with nonviscous liquid. The phase velocities calculated by the thin-shell approach were compared with those calculated by the thick-cylinder approach. In contrast to the case of the empty pipe, where only two modes were obtained and the first mode was calculated in a limited frequency range, the results for the liquid-filled pipe exhibits a large number of modes due to the large number of branches of the apparent liquid mass. Several of the lowest modes of the waves in a liquid-filled pipe were calculated for various pipe thicknesses in a low frequency range. The thin-shell approach was valid for a reasonable range of pipe thicknesses.

**Key Words:** Elastic Wave, Axisymmetric Mode, Phase Velocity, Frequency, Shell, Pipe

## 1. Introduction

Axisymmetric longitudinal waves in an elastic pipe have been used to inspect pipelines (Rose, 2002) and also to measure fluid properties in a pipe (Kim et al., 2003a ; Hwang and Kim, 2004). In order to predict the propagation of elastic waves and to compare with wave experiments for a pipe with a flowing fluid, the development of a simplified wave theory is desired in order to investigate the fluid flow effect. Experiments for guided waves are often reported (Ahn and Nam, 2003 ; Kim et al., 2003b).

A fully-elastic approach, which is referred to as a thick-cylinder approach hereafter, for axisymmetric wave propagation has already been developed for an empty cylinder (Gazis, 1959) and for a liquid-filled cylinder (Sinha et al., 1992 ; Cho and Rose, 1996). Meanwhile, a thin-shell approach has been developed as a simplified tool for the analysis of wave propagation in a cylinder. For an empty cylinder, even non-axisymmetric wave propagation has been described by the thin-shell approach (Junger and Feit, 1986 ; Graff, 1991). Characteristics of wave propagation and energy distribution have been derived for a cylinder with internal stationary liquid (Fuller and Fahy, 1982) and for a cylinder with internal liquid flow (Brevart and Fuller, 1993). The phase velocity of the axisymmetric wave in a water-filled cylinder has been calculated from the equations derived by the thin-shell approach (Kim et al., 2003), and it was compared with measurement (Hwang and Kim, 2004).

The thin-shell approach is known to yield reasonable results as long as the wall thickness is small relative to the radius and the wavelength is large compared to the wall thickness. It is necessary to find the range of parameters for which the

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thin-shell approach for the analysis of elastic wave propagation in a pipe is valid. Here, the validity and limitation of the thin-shell approach is presented by comparing the phase velocities calculated by this approach with those calculated by the thick-cylinder approach.

The major approximations in the analysis were as follows: (1) the pipe is perfectly circular, concentric, and infinitely long, (2) the liquid in the pipe is ideal, that is inviscid, compressible, irrotational, and stationary, (3) transverse shear stresses and bending and twisting moments are neglected, and (4) the pipe is not pre-stressed, that is, static stresses due to liquid pressure, pipe weight, and mounting are neglected, and all stresses in the pipe are the result of the induction of the elastic waves.

## 2. Thick-Cylinder Approach

In order to compare with the thin-shell approach, the equations of the thick-cylinder approach are briefly reviewed here. Consider elastic waves propagating in a pipe having an inner radius $a$ and an outer radius $b$ as shown in Fig. 1. The pipe has a wall thickness $h=b-a$ and a mean radius $R=(a+b)/2$.

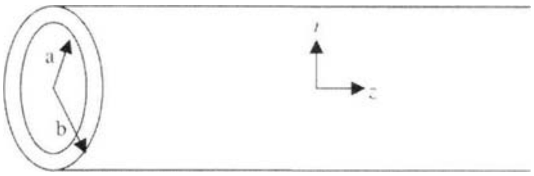


**Fig. 1** Schematic diagram of a pipe showing radii and coordinates

### 2.1 Analysis procedure

Radial and axial displacements $u_r$ and $u_z$ in a pipe and $w_r$ and $w_z$ in a liquid are used to describe the axisymmetric motion of the waves. Normal stress $\sigma_r$ and shear stress $\tau_{rz}$ in the pipe are expressed in terms of the displacement components $u_r$ and $u_z$ as follows (Achenbach, 1975):

$$\sigma_r=\lambda\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right)+2G\frac{\partial u_r}{\partial r} \tag{1a}$$

$$\tau_{rz}=G\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) \tag{1b}$$

Here, $\lambda$ and $G$ are the Lame elastic constants and $G$ is the shear modulus. Similarly, normal stress $\sigma_w$ in the liquid is expressed in terms of the displacement components $w_r$ and $w_z$ as follows:

$$\sigma_w=\lambda_w\left(\frac{\partial w_r}{\partial r}+\frac{w_r}{r}+\frac{\partial w_z}{\partial z}\right) \tag{1c}$$

Coupling of the variables in the equations of motion can be avoided by introducing displacement potentials $\phi$, $\psi_\theta$, and $\phi_w$, which are related to the displacement components as follows (Achenbach, 1975):

$$u_r=\frac{\partial\phi}{\partial r}-\frac{\partial\psi_\theta}{\partial z} \tag{2a}$$

$$u_z=\frac{\partial\phi}{\partial z}+\frac{\partial\psi_\theta}{\partial r}+\frac{\psi_\theta}{r} \tag{2b}$$

$$w_r=\frac{\partial\phi_w}{\partial r} \tag{2c}$$

$$w_z=\frac{\partial\phi_w}{\partial z} \tag{2d}$$

For the waves traveling in the $z$ direction, harmonic wave motion is assumed to have the solution of the following form:

$$\phi(r,z,t)=\Phi(r)\exp[i(kz-\omega t)] \tag{3a}$$

$$\psi_\theta(r,z,t)=\Psi(r)\exp[i(kz-\omega t)] \tag{3b}$$

$$\phi_w(r,z,t)=\Phi_w(r)\exp[i(kz-\omega t)] \tag{3c}$$

where $\omega$ is the circular frequency and $k$ is the wavenumber.

The uncoupled equations of motion are rewritten as follows:

$$\Phi''+\frac{1}{r}\Phi'+q_L^2\Phi=0 \quad q_L^2=\frac{\omega^2}{c_L^2}-k^2 \tag{4a}$$

$$\Psi''+\frac{1}{r}\Psi'+\left(q_T^2-\frac{1}{r^2}\right)\Psi=0 \quad q_T^2=\frac{\omega^2}{c_T^2}-k^2 \tag{4b}$$

$$\Phi_w''+\frac{1}{r}\Phi_w'+q_w^2\Phi_w=0 \quad q_w^2=\frac{\omega^2}{c_w^2}-k^2 \tag{4c}$$

where $c_L=\sqrt{(\lambda+2G)/\rho}$ and $c_T=\sqrt{G/\rho}$ are, respectively, the velocities of longitudinal and transverse waves in pipe material and $c_w$ is the velocity of a longitudinal wave in liquid. Here $\rho$ represents the mass density. Equations (4a)-(4c) are typical Bessel equations and their solutions have the following form :

$$\Phi(r) = B_1 J_0(q_L r) + B_2 Y_0(q_L r) \qquad (5a)$$

$$\Psi(r) = B_3 J_1(q_T r) + B_4 Y_1(q_T r) \qquad (5b)$$

$$\Phi_w(r) = B_5 J_0(q_w r) \qquad (5c)$$

The following boundary conditions are applied on the outer and inner surface of the pipe :

$$\text{at } r=b,\ \sigma_r=0 \text{ and } \tau_{rz}=0 \qquad (6a,b)$$

$$\text{at } r=a,\ \sigma_r=\sigma_w,\ \tau_{rz}=0, \text{ and } u_r=w_r \qquad (6c,d,e)$$

Finally a system of equations is obtained as follows (Sinha, 1992 ; Cho and Rose, 1996 ; Rose, 1999) :

$$[D_{ij}]\{B_j\}=\{0\} \quad i, j=1, 2, \cdots, 5 \qquad (7)$$

where $\{B_j\}$ consists of the unknown constants in Eq. (5) and the elements $D_{ij}$ of the matrix $[D_{ij}]$ are listed in the Appendix. When the pipe is empty, the equations are simplified with $i, j=1, 2, 3, 4$.

### 2.2 Phase velocity

For nontrivial solutions of the unknown amplitude $B_j$, the elements $D_{ij}$ of the matrix $[D_{ij}]$ must satisfy the following characteristic equation :

$$\det|D_{ij}|=0 \qquad (8)$$

The solution of Eq. (8) is the wavenumber $k$, which may be a complex number $(k=k_R+ik_I)$. The imaginary part $k_I$ represents the attenuation. The phase velocity $c$ is obtained from the real part $k_R$ as :

$$c=\frac{\omega}{k_R} \qquad (9)$$

As a demonstration, phase velocity of the wave in an aluminum pipe was calculated with material properties listed in Table 1 and displayed as a function of frequency for various values of thickness-to-radius ratio $h/R$ in Fig. 2. The phase velocity $c$ was normalized by the velocity $c_b=\sqrt{E/\rho}$ of a longitudinal wave in a thin bar and the frequency $\omega$ was normalized by $c_b/R$. The numbers 1, 2, 3, and 4 in the graph mean the modes. The curves show the well-known fact that the phase velocity $c$ approaches the transverse wave velocity $c_T$ as the frequency $\omega$ goes to infinity. The focus here is to understand the dispersion curves at low frequency, and modes 1 and 2 are considered in the next sections. It is noted that at low frequency the phase velocity of the axisymmetric longitudinal wave shows little difference according to the thickness of the pipe.

**Table 1** Elastic properties of aluminum (1100-H14) and wave velocities calculated from the elastic properties

| Property | | Value |
|---|---|---|
| Elastic property | Mass density, $\rho$ | 2,710 kg/m$^3$ |
| | Young's modulus, $E$ | 70 GPa |
| | Shear modulus, $G$ | 26 GPa |
| | Lame constant, $\lambda$ | 58.5 GPa |
| | Poisson's ratio, $\nu$ | 0.346 |
| Wave velocity | Longitudinal wave, $c_L$ | 6,385 m/s |
| | L-wave in a plate, $c_p$ | 5,417 m/s |
| | L-wave in a bar, $c_b$ | 5,082 m/s |
| | Transverse wave, $c_T$ | 3,097 m/s |

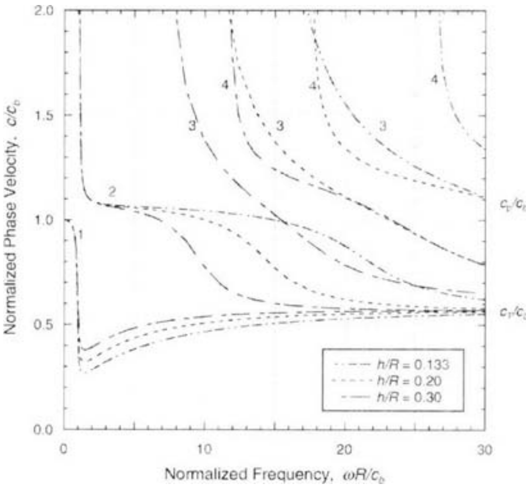


**Fig. 2** Dispersion curves of the longitudinal waves in an empty pipe, calculated for various $h/R$ ratios by the thick-cylinder approach

## 3. Thin-Shell Approach

For the pipe shown in Fig. 1, the pipe wall is treated as a thin shell. Shell theory has the benefit of providing simpler expressions than the ones generated using the full equations of linear elasticity described in the previous section.

### 3.1 Modeling

In the special case of an axisymmetric, cylindrical membrane shell, the general thin-shell theory (Junger and Feit, 1986) is simplified by eliminating the terms representing circumferential variations to:

$$\frac{\partial^2 u_z}{\partial z^2}+\frac{\nu}{R}\frac{\partial u_r}{\partial z}=\frac{1-\nu^2}{c_b^2}\frac{\partial^2 u_z}{\partial t^2} \tag{10}$$

and

$$-\frac{u_r}{R^2}-\frac{\nu}{R}\frac{\partial u_z}{\partial z}+\frac{q(1-\nu^2)}{\rho h c_b^2}=\frac{1-\nu^2}{c_b^2}\frac{\partial^2 u_r}{\partial t^2} \tag{11}$$

In the above $q$ is the radial load and $\nu$ is the Poisson's ratio of the pipe material.

The liquid pressure $p$ satisfies the wave equation (Kinsler et al., 2000):

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c_w^2}\frac{\partial^2 p}{\partial t^2} \tag{12}$$

where $c_w$ is the sound velocity in the liquid medium. The radial pressure $q$ in Eq. (11) equals the pressure $p$ at $r=a$, that is:

$$q=p|_{r=a} \tag{13}$$

Once the pressure equation is solved, the radial liquid velocity $v_r$ can be computed utilizing the linearized radial momentum equation:

$$\frac{\partial v_r}{\partial t}=-\frac{1}{\rho_w}\frac{\partial p}{\partial r} \tag{14}$$

At the surface of a pipe:

$$\left.\frac{\partial v_r}{\partial t}\right|_{r=a}=\frac{\partial^2 u_r}{\partial t^2} \tag{15}$$

Equations (14) and (15) are combined to obtain the relation between $u_r$ and $p$:

$$\left.\frac{\partial p}{\partial r}\right|_{r=a}=-\rho_w\frac{\partial^2 u_r}{\partial t^2} \tag{16}$$

### 3.2 Solutions

Solutions for the wave propagation are sought as the following forms:

$$u_r(z,t)=\bar{U}_r\exp[i(kz-\omega t)] \tag{17a}$$

$$u_z(z,t)=\bar{U}_z\exp[i(kz-\omega t)] \tag{17b}$$

and

$$p(r,z,t)=P(r)\exp[i(kz-\omega t)] \tag{17c}$$

Upon substituting Eq. (17c) in Eq. (12), a typical Bessel equation is obtained for the pressure amplitude $P(r)$.

$$\frac{d^2P}{dr^2}+\frac{1}{r}\frac{dP}{dr}+q_w^2P=0 \tag{18}$$

where $q_w$ was defined in Eq. (4c).

Equation (18) admits a solution of the form:

$$P(r)=C_1J_0(q_w r)+C_2Y_0(q_w r) \tag{19}$$

In Eq. (19) $J_0$ and $Y_0$ are Bessel functions of the first kind and of the order 0. $J_0(0)$ is finite while $Y_0(0)$ is infinite. Thus, to assure bounded solutions at $r=0$, $C_2$ is set to 0.

Since

$$\begin{aligned}\left.\frac{dP}{dr}\right|_{r=a}&=C_1\left.\frac{dJ_0(q_w r)}{dr}\right|_{r=a}\\&=-C_1q_wJ_1(q_w a)\end{aligned} \tag{20}$$

the boundary condition (16) can be rewritten as:

$$\begin{aligned}p|_{r=a}&=C_1J_0(q_w a)\exp[i(kz-\omega t)]\\&=\rho_w\frac{J_0(q_w a)}{q_wJ_1(q_w a)}\frac{\partial^2 u_r}{\partial t^2}\end{aligned} \tag{21}$$

Equation (21) includes the apparent mass of the liquid per unit area (Kim et al., 2003b):

$$M(q_w a)=-\rho_w\frac{J_0(q_w a)}{q_wJ_1(q_w a)} \tag{22}$$

Upon expressing the liquid pressure in terms of the radial displacement of the wall, Eq. (11) can be rewritten as:

$$-\frac{u_r}{R^2}-\frac{\nu}{R}\frac{\partial u_z}{\partial z}=\left(1+\frac{M(q_w a)}{\rho h}\right)\frac{1-\nu^2}{c_b^2}\frac{\partial^2 u_r}{\partial t^2} \tag{23}$$

The above expression suggests that the effect of the liquid increases as the solid density $\rho$ and the shell thickness $h$ decreases.

Upon substituting Eqs. (17a) and (17b) into Eqs. (10) and (23) and assuming $a\approx R$, one obtains:

$$\left[k^2-(1-\nu^2)\left(\frac{\omega}{c_b}\right)^2\right]\bar{U}_z-ik\frac{\nu}{R}\bar{U}_r=0 \quad (24a)$$

$$ik\frac{\nu}{R}\bar{U}_z+\left[\frac{1}{R^2}-\left(1+\frac{M(q_wR)}{\rho h}\right)(1-\nu^2)\left(\frac{\omega}{c_b}\right)^2\right]\bar{U}_r=0 \quad (24b)$$

### 3.3 Phase velocity

In general, the axial wavenumber $k$ can be a complex number, and the imaginary part represents the attenuation. When a pipe is surrounded with a liquid, the elastic wave in the pipe is leaky and the attenuation is not negligible. Since this work considers a liquid confined in a pipe, $k$ is assumed to be real, and $k=\omega/c$. Requiring the equations to admit non-trivial solutions, one can obtain a characteristic equation :

$$\begin{aligned}&1-\left(1+\frac{M(q_wR)}{\rho h}\right)\left(\frac{\omega R}{c_b}\right)^2\\&-\left[1-\left(1+\frac{M(q_wR)}{\rho h}\right)(1-\nu^2)\left(\frac{\omega R}{c_b}\right)^2\right]\left(\frac{c}{c_b}\right)^2=0\end{aligned} \quad (25)$$

Equation (25) represents the dispersion relation between the normalized phase velocity $c/c_b$ and the normalized frequency $\omega R/c_b$.

When the pipe is in vacuum (or in air), $M=0$ and Eq. (25) can be solved explicitly to obtain the normalized phase velocity $c/c_b$ as a function of the normalized frequency $\omega R/c_b$ :

$$\frac{c}{c_b}=\sqrt{\frac{1-(\omega R/c_b)^2}{1-(1-\nu^2)(\omega R/c_b)^2}} \quad (26)$$

## 4. Results and Discussion

Based on the solutions derived by the thick-cylinder and by the thin-shell approaches, phase velocities of the axisymmetric longitudinal waves were calculated for an empty pipe and for a water-filled pipe. The material properties of the aluminum pipe (Beer and Johnston, 1995) are listed in Table 1 with water density of 998 kg/m$^3$ and wave velocity $c_w$ in water at 1,481 m/s.

### 4.1 For an empty pipe

The phase velocity $c$ of a longitudinal wave in an empty pipe was calculated from Eqs. (8) and (9) for the thick-cylinder approach and from Eq. (26) for the thin-shell approach. The phase velocity $c$ normalized by the propagation velocity $c_b$ of a longitudinal wave in a thin bar was displayed in Fig. 3 as a function of the normalized frequency $\omega R/c_b$. The numbers 1 and 2 in the graph mean the modes. In Fig. 3, dashed lines present the dispersion curves for various $h/R$ ratios obtained by the thick-cylinder approach, and the solid lines represent the dispersion curves obtained by the thin-shell approach. Comparison of the two results reveals that the thin-shell approach is valid within a limited range of frequency. Thin-shell approach yields only two modes.

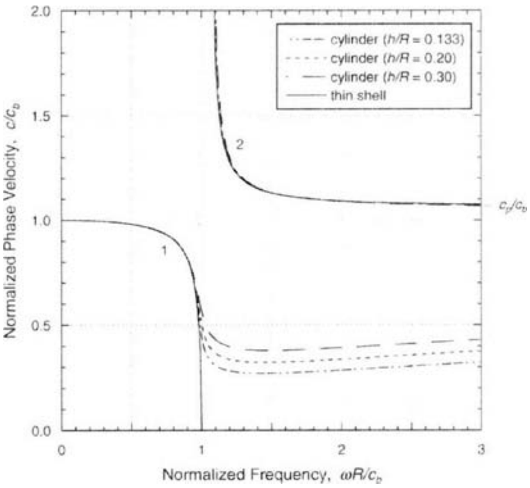


**Fig. 3** Comparison of the dispersion curves calculated by the thick-cylinder approach and by the thin-shell approach for an empty pipe

The first mode calculated by the thin-shell approach is valid when the frequency $\omega$ is less than the thin-bar wave velocity $c_b$ divided by the mean radius $R$ of the pipe. The phase velocity $c$ of the first mode approaches $c_b$ as the frequency or the pipe radius decreases to zero. It is understood from the comparison that when the frequency or the pipe radius is small $(\omega R/c_b<1)$ the thickness deformation of the pipe is negligible as if it were a thin shell. It is also understood that when the frequency or the pipe radius is very small $(\omega R/c_b\ll 1)$ the wave motion in the pipe is similar to that in a thin bar.

The second mode calculated by the thin-shell approach is valid when the frequency $\omega$ is larger than the thin-bar wave velocity $c_b$ divided by the mean radius $R$ of the pipe. As observed in Eq. (26), it is shown that the cut-off frequency of the second mode is $c_p/R$, where $c_p=\sqrt{E/\rho(1-\nu^2)}$ is the propagation velocity of a longitudinal wave in a thin plate. The phase velocity of the second mode approaches $c_p$, as the frequency or the pipe radius becomes very large, which is anticipated from Eq. (26). As shown in Fig. 2, the phase velocity of the second mode maintains at $c_p$ in the range of low frequency and drops to $c_T$ in higher frequency. The second mode calculated by the thin-shell approach is valid in the frequency range where the phase velocity is maintained at $c_p$. Finally, the thin-shell approach does not yield the dispersion curves of the modes higher than the second.

### 4.2 For a water-filled pipe

The phase velocity $c$ of a longitudinal wave in a water-filled pipe was calculated from Eqs. (8) and (9) for the thick-cylinder approach and from Eq. (25) for the thin-shell approach. The phase velocity $c$ normalized by the propagation velocity $c_b$ of a longitudinal wave in a thin bar was displayed in Fig. 4 as a function of the normalized frequency $\omega R/c_b$. The numbers 1 and 2 in the graph mean the modes and 2a, 2b, and 2c mean the branches of the second mode. In Fig. 4, dashed lines represent the dispersion curves for various $h/R$ ratios obtained by the thick-cylinder approach, and the solid lines are the dispersion curves obtained by the thin-shell approach. Comparison of the two results reveal that the thin-shell approach is valid within a limited range of frequency.

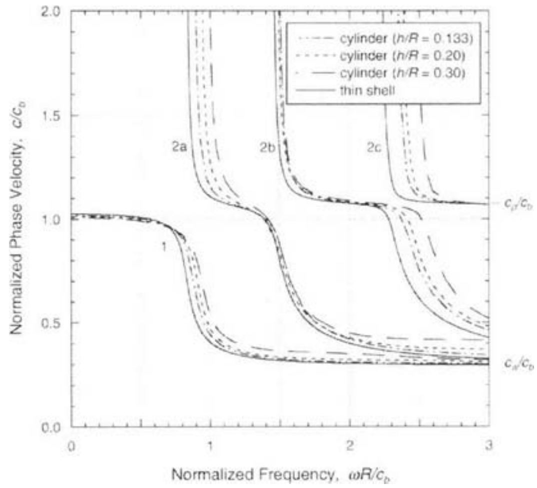


**Fig. 4** Comparison of the dispersion curves calculated by the thick-cylinder approach and by the thin-shell approach for a pipe containing stationary water

In contrast to the case of the empty pipe (Fig. 3), Fig. 4 exhibits a large number of modes. The presence of a large number of modes could be anticipated based on the large number of branches of the apparent liquid mass in Eq. (22) (Kim et al., 2003b). The phase velocities of the second and higher modes approach $c_p$ in some frequency range and then $c_w$ in higher frequency range.

## 5. Conclusion

The phase velocities of the axisymmetric longitudinal wave propagating in an elastic pipe were calculated by the thin-shell approach and compared with those calculated by the thick-cylinder approach. The two results showed good agreements for the first mode of the wave in the circular frequency range less than the longitudinal thin-bar wave velocity divided by the mean radius of the pipe. The two results for the second mode also showed good agreement in the range from the cut-off frequency up to the frequency where the phase velocity is maintained at the longitudinal thin-plate velocity. The validity and limitation of the thin-shell approach was quantitatively investigated in this paper.

In contrast to the case of the empty pipe, where only two modes were obtained and the first mode was calculated in a limited frequency range, the results for the liquid-filled pipe exhibits a large number of modes due to the large number of branches of the apparent liquid mass. Several of the lowest modes of the waves in a liquid-filled pipe were calculated for various pipe thicknesses in a wide frequency range. In the low frequency range, the thin-shell approach is valid for reasonable pipe thicknesses when considering the effect of a stationary liquid on the wave propagation in a pipe. The results of this paper support the possibility of using the thin-shell approach in the theoretical prediction of the effect of fluid flow on the wave propagation in a pipe.

## Acknowledgments

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## Appendix

The elements of the matrix $[D_{ij}]$ in Eq. (8) are as follows:

$$D_{11}=-[\lambda(q_L^2+k^2)+2Gq_L^2]J_0(q_Lb)+2G\frac{q_L}{b}J_1(q_Lb)$$

$$D_{12}=-[\lambda(q_L^2+k^2)+2Gq_L^2]Y_0(q_Lb)+2G\frac{q_L}{b}Y_1(q_Lb)$$

$$D_{13}=-2iGk\left[q_TJ_0(q_Tb)-\frac{1}{b}J_1(q_Tb)\right]$$

$$D_{14}=-2iGk\left[q_TY_0(q_Tb)-\frac{1}{b}Y_1(q_Tb)\right]$$

$$D_{15}=0$$

$$D_{21}=-2iGkq_LJ_1(q_Lb)$$

$$D_{22}=-2iGkq_LY_1(q_Lb)$$

$$D_{23}=G(k^2-q_T^2)J_1(q_Lb)$$

$$D_{24}=G(k^2-q_T^2)Y_1(q_Tb)$$

$$D_{25}=0$$

$$D_{31}=-[\lambda(q_L^2+k^2)+2Gq_L^2]J_0(q_La)+2G\frac{q_L}{a}J_1(q_La)$$

$$D_{32}=-[\lambda(q_L^2+k^2)+2Gq_L^2]Y_0(q_La)+2G\frac{q_L}{a}Y_1(q_La)$$

$$D_{33}=-2iGk\left[q_TJ_0(q_Ta)-\frac{1}{a}J_1(q_Ta)\right]$$

$$D_{34}=-2iGk\left[q_TY_0(q_Ta)-\frac{1}{a}Y_1(q_Ta)\right]$$

$$D_{35}=\lambda_w(q_w^2+k^2)J_0(q_wa)$$

$$D_{41}=-2iGkq_LJ_1(q_La)$$

$$D_{42}=-2iGkq_LY_1(q_La)$$

$$D_{43}=G(k^2-q_T^2)J_1(q_Ta)$$

$$D_{44}=G(k^2-q_T^2)Y_1(q_Ta)$$

$$D_{45}=0$$

$$D_{51}=-q_LJ_1(q_La)$$

$$D_{52}=-q_LY_1(q_La)$$

$$D_{53}=-ikJ_1(q_Ta)$$

$$D_{54}=-ikY_1(q_Ta)$$

$$D_{55}=q_wJ_1(q_wa)$$

where $q_L$, $q_T$, and $q_w$ were defined in Eq. (4).

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